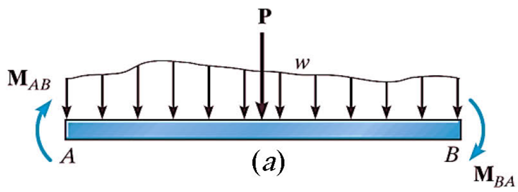
9 DISPLACEMENT METHOD OF ANALYSIS: MOMENT DISTRIBUTION

9.1 Moment Distribution

Moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession، the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply.

Sign Convention.

We will establish the same sign convention as that established for the slope-deflection equations: ***Clockwise moments*** that act ***on the member*** are considered ***positive***, whereas ***counterclockwise moments*** are ***negative***, **Fig. *a***.

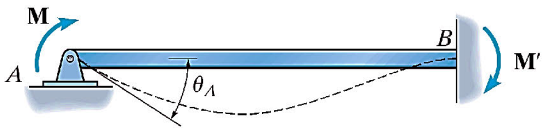


Fixed-End Moments (*FEMs*).

The moments at the "walls" or fixed joints of a loaded member are called ***fixed-end moments***. These moments can be determined from the **table (8-1)**, depending upon the type of loading on the member.

Member Stiffness Factor.

The amount of moment required to rotate the end A of the beam by $\boldsymbol{\theta} = \mathbf{1}$ **rad**

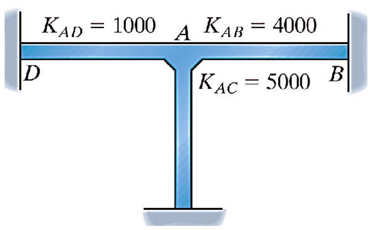


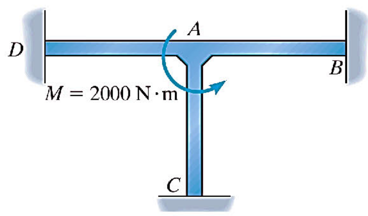
$$M = \frac{4EI\theta}{L}$$

$$\boldsymbol{K = \frac{4EI}{L}} \quad \text{Far End Fixed}$$

Joint Stiffness Factor.

If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the ***total stiffness factor*** at the joint is the sum of the member stiffness factors at the joint, that is, $\boldsymbol{K_T = \sum K}$**.**

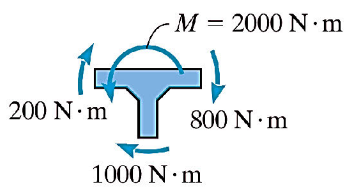




$$M = M_{AD} + M_{AB} + M_{AC}$$

$$K_t = \sum K_i$$

$$K_t = 4000 + 5000 + 1000 = 10\ 000.$$



Distribution Factor (*DF*).

If a moment ***M*** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the ***distribution factor*** (**DF**).

$$M = M_1 + M_2 + M_3 + \ldots$$

$$M = K_1\theta + K_2\theta + K_3\theta + .. = \theta\sum K$$

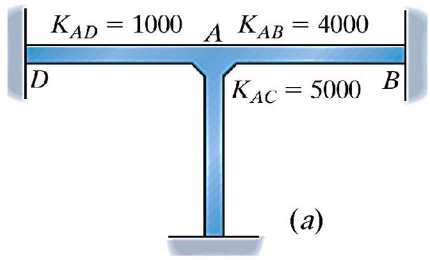
$$DF = \frac{M_i}{M} = \frac{\theta K_i}{\theta\sum K} = \frac{K_i}{\sum K} \quad \Rightarrow \quad \boldsymbol{DF = \frac{K_i}{\sum K}}$$

For example, the distribution factors for members ***AB***, ***AC***, and ***AD*** at joint ***A*** in **Fig. *a*** are,

$$DF_{AB} = \frac{K_i}{\sum K} = \frac{4000}{10000} = 0.4$$

$$DF_{AC} = \frac{K_i}{\sum K} = \frac{5000}{10000} = 0.5$$

$$DF_{AD} = \frac{K_i}{\sum K} = \frac{1000}{10000} = 0.1$$



(*a*)

Member Relative-Stiffness Factor.

For the same material, the term **4*E*** of the stiffness factor will cancel then it is easier to determine the relative stiffness factor as:

$$\boldsymbol{K_R = \frac{I}{L}} \quad \text{Far End Fixed}$$

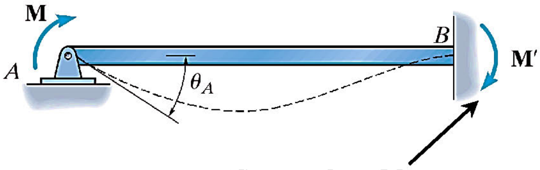
and use this for the computations of the ***DF*.**

Carry-Over Factor.

The fraction of the moment that is carried out from the joint to the end:

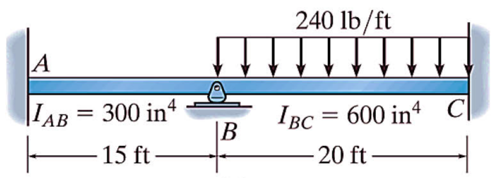
$$\boldsymbol{M' = \frac{1}{2}M}$$

in the case of a beam with ***the far end fixed***, the carry-over factor is +½. The plus sign indicates both moments act in the same direction.



EXAMPLE 9.1.1

Use moment distribution method to determine the moment at joint ***A***, ***B***, and ***C***, for the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

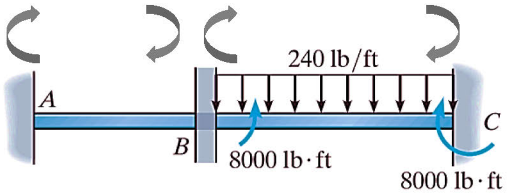
Stiffness Factor:

$$K_R = \frac{I}{L}$$

$$K_{AB} = \frac{300}{15} = 20 \qquad , K_{BC} = \frac{600}{20} = 30$$

$$K_{AB} : K_{BC}$$

$$2 : 3$$



joint *B* held fixed
(b)

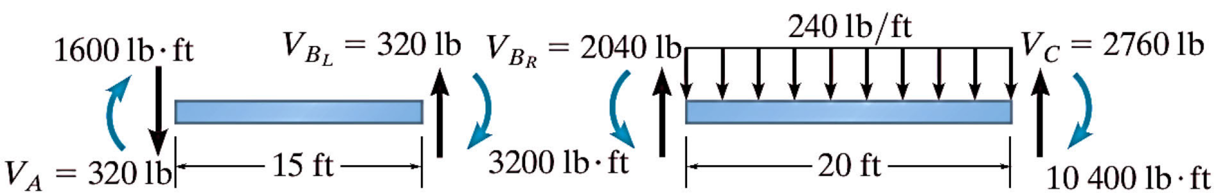
Distribution Factor:

$$DF_{AB} = \frac{2}{5} = 0.4 \;, \qquad DF_{BC} = \frac{3}{5} = 0.6$$

Fixed-End Moments (FEMs):

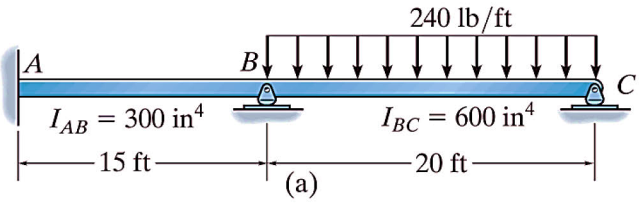
$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000 \text{ lb.ft} \;, \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = \quad 8000 \text{ lb.ft}$$

Joint	***A***	***B***		***C***
Member	***AB***	***BA***	***BC***	***CB***
DF	**0**	**0.4**	**0.6**	**0**
FEM			**-8000**	**8000**
Dist. CO.		**3200**	**4800**	
Dist. CO.	**1600**			**2400**
∑*M*	**1600**	**3200**	**-3200**	**10400**



EXAMPLE 9.1.2

Use moment distribution method to determine the moment at joint *A*, *B*, and *C*, for the beam shown in **Fig. *a***. ***EI*** is constant.



Solution

Stiffness Factor:

$$K_R = \frac{I}{L}$$

$$K_{AB} = \frac{300}{15} = 20 \quad , K_{BC} = \frac{600}{20} = 30 \quad \Rightarrow K_{AB} : K_{BC} = 2:3$$

Distribution Factor:

$$DF_{AB} = \frac{2}{5} = 0.4 \, , \qquad DF_{BC} = \frac{3}{5} = 0.6$$

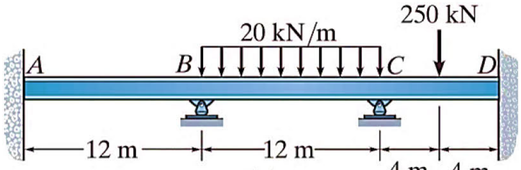
Fixed-End Moments (FEMs):

$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000 \text{ lb.ft} \, , \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000 \text{ lb.ft}$$

Joint	*A*	*B*		*C*
Member	***AB***	***BA***	***BC***	***CB***
DF	**0**	**0.4**	**0.6**	**1**
FEM			**-8000**	**8000**
Dist. CO.		**3200**	**4800**	**-8000**
FEM	**1600**		**-4000**	**2400**
Dist. CO.		**1600**	**2400**	**-2400**
FEM	**800**		**-1200**	**1200**
Dist. CO.		**480**	**720**	**-1200**
FEM	**240**		**-600**	**360**
Dist. CO.		**240**	**360**	**-360**
FEM	**120**		**-180**	**180**
Dist. CO.		**72**	**108**	**-180**
FEM	**36**		**-90**	**54**
Dist. CO.		**36**	**54**	**-54**
FEM	**18**		**-27**	**27**
Dist. CO.		**10.8**	**16.2**	**-27**
FEM	**5.4**		**-13.5**	**8.1**
Dist. CO.		**5.4**	**8.1**	**-8.1**
FEM	**2.7**		**-4.05**	**4.05**
Dist. CO.		**1.62**	**2.43**	**-4.05**
FEM	**0.81**		**-2.025**	**1.22**
Dist. CO.		**0.81**	**1.22**	**-1.22**
FEM	**0.40**		**-0.61**	**0.61**
Dist. CO.		**0.244**	**0.37**	**-0.61**
$\sum M$	**2823.31**	**5646.87**	**-5646.87**	**0**

EXAMPLE 9.1.3

Determine the internal moments at each support of the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

Members *AB* & *BC*:

$$K_{BA}=\frac{I}{12},\quad K_{BC}=\frac{I}{12}$$

$$K_{BA}:K_{BC}=1:1$$

$$DF_{BA}=\frac{1}{2}=0.5$$

$$DF_{BC}=\frac{1}{2}=0.5$$

Member *BC* & *CD*:

$$K_{CB}=\frac{I}{12},\quad K_{CD}=\frac{I}{8}$$

$$K_{CB}:K_{CD}=8:12=2:3$$

$$DF_{CB}=\frac{2}{5}=0.4$$

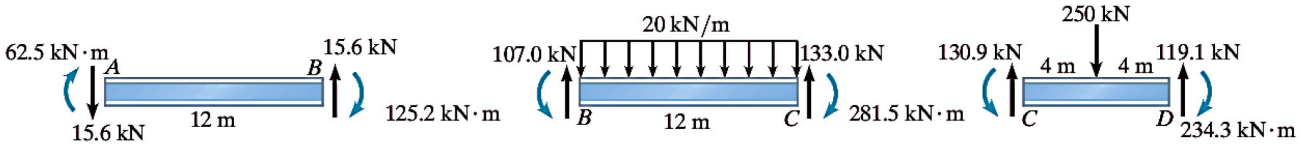
$$DF_{CD}=\frac{3}{5}=0.6$$

Fixed-End Moments (FEMs):

$$(FEM)_{BC}=\frac{wL^2}{12}=-\frac{20(12)^2}{12}=-240\text{ kN.m},\qquad (FEM)_{CB}=\frac{wL^2}{12}=\frac{20(12)^2}{12}=240\text{ kN.m}$$

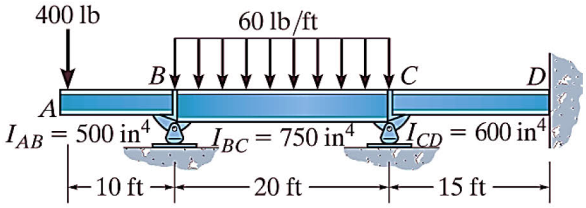
$$(FEM)_{CD}=\frac{PL}{8}=-\frac{250(8)}{8}=-250\text{ kN.m},\qquad (FEM)_{DC}=\frac{PL}{8}=\frac{250(8)}{8}=250\text{ kN.m}$$

Joint	A	B		C		D
Member	***AB***	***BA***	***BC***	***CB***	***CD***	***DC***
DF	**0**	**0.5**	**0.5**	**0.4**	**0.6**	**0**
FEM			**-240**	**240**	**-250**	**250.00**
Dist. CO.		**120**	**120**	**4**	**6**	
FEM	**60**		**2**	**60**		**3.00**
Dist. CO.		**-1**	**-1**	**-24**	**-36**	
FEM	**-0.5**		**-12**	**-0.5**		**-18.00**
Dist. CO.		**6**	**6**	**0.2**	**0.3**	
FEM	**3**		**0.1**	**3**		**0.15**
Dist. CO.		**-0.05**	**-0.05**	**-1.2**	**-1.8**	
FEM	**-0.025**		**-0.6**	**-0.025**		**-0.90**
Dist. CO.		**0.3**	**0.3**	**0.01**	**0.015**	
$\sum M$	**62.475**	**125.25**	**-125.25**	**281.485**	**-281.485**	**234.25**



EXAMPLE 9.1.3

Determine the internal moments at each support of the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

Member *BC* & *CD*:

$$K_{CB} = \frac{750}{20} = 37.5\ , \qquad K_{CD} = \frac{600}{15} = 40\ , \qquad K_{CB} : K_{CD} = 37.$$

$$DF_{CB} = \frac{37.5}{77.5} = 0.484\ , \quad DF_{CD} = \frac{40}{77.5} = 0.516$$

Fixed-End Moments (FEMs):

Due to the overhang, $(FEM)_{BA}$ = 400 lb (10 ft) = 4000 lb.ft

$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000 \text{ lb.ft}\ , \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000 \text{ lb.ft}$$

Joint	***B***		***C***		***D***
Member	***BA***	***BC***	***CB***	***CD***	***DC***
DF	**0**	**1**	**0.484**	**0.516**	**0**
FEM	**4000**	**-2000**	**2000**		
Dist. CO.		**-2000**	**-968**	**-1032**	
FEM		**-484**	**-1000**		**-516**
Dist. CO.		**484**	**484**	**516**	
FEM		**242**	**242**		**258**
Dist. CO.		**-242**	**-117.13**	**-124.87**	
FEM		**-58.56**	**-121**		**-62.44**
Dist. CO.		**58.56**	**58.56**	**62.44**	
FEM		**29.28**	**29.28**		**31.22**
Dist. CO.		**-29.28**	**-14.17**	**-15.11**	
FEM		**-7.09**	**-14.64**		**-7.55**
Dist. CO.		**7.09**	**7.09**	**7.55**	
FEM		**3.54**	**3.54**		**3.78**
Dist. CO.		**-3.54**	**-1.71**	**-1.83**	
FEM		**-0.86**	**-1.77**		**-0.91**
Dist. CO.		**0.86**	**0.86**	**0.91**	
FEM		**0.43**	**0.43**		**0.46**
Dist. CO.		**-0.43**	**-0.21**	**-0.22**	
FEM		**-0.10**	**-0.21**		**-0.11**
Dist. CO.		**0.10**	**0.10**	**0.11**	
$\sum M$	**4000**	**-4000**	**587.02**	**-587.02**	**-43.56**